Second-order differential equation with linear dependence in the solutions

Solve $y'' + 4y' + 4y = 2e^{-2x}$

Solution

We solve the homogeneous part with the characteristic equation:

$$r^2 + 4r + 4 = 0$$

From here we find that the root is $r = -2$

$$y_H = C_1 e^{-2x} + xC_2 e^{-2x}$$

For the particular solution, we propose $Ke^{-2x}x^2$ since if we do not multiply by x^2, the solution is linearly dependent with a part of the homogeneous solution. We calculate the derivatives:

$$y_C' = 2xKe^{-2x} - 2Kx^2e^{-2x}$$

$$y_C'' = 2Ke^{-2x} - 4xKe^{-2x} - 4Kxe^{-2x} + 4Kx^2e^{-2x} = 2Ke^{-2x} - 8xKe^{-2x} + 4Kx^2e^{-2x}$$

We solve:

$$2Ke^{-2x} - 8xKe^{-2x} + 4Kx^2e^{-2x} + 4(2xKe^{-2x} - 2Kx^2e^{-2x}) + 4Ke^{-2x}x^2 = 2e^{-2x}$$

$$2Ke^{-2x} - 8xKe^{-2x} + 4Kx^2e^{-2x} + 8xKe^{-2x} - 8Kx^2e^{-2x} + 4Ke^{-2x}x^2 = 2e^{-2x}$$

$$2Ke^{-2x} = 2e^{-2x}$$

From this we get:

$$2K = 2$$

Therefore $K = 1$

$$y_C = x^2e^{-2x}$$

The general solution is:

$$y_g = y_H + y_C = C_1e^{-2x} + xC_2e^{-2x} + x^2e^{-2x}$$